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On Solutions of Some Nonlinear Recurrences

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We investigate when non-linear recurrences in vector lattices (that is, partially ordered sets which are vector spaces, and in which every two elements have a maximum and a minimum) have unique positive solutions. As an application of these techniques we study positive solutions of real recurrences which arise in some problems in approximation theory. A typical recurrence of this type is $n/a_n = a_{n+1} + a_n + a_{n-1}$ for $n \geqslant 1$ with $a_0 = 0$, and the question is whether there is a solution with $a_n > 0$ for all $n \geqslant 1$. © 1987 Academic Press, Inc.

1. INTRODUCTION

The object of this paper is to give conditions which ensure that there is a unique solution to certain non-linear recurrences in vector lattices. This has applications to recurrences which arise in the study of orthogonal polynomials ([2, 6]) where the lattice is $\mathbb{R}$. When the lattice is a function space, it has applications to iterative solutions of differential and integral equations (see [1, 7]). However, here we shall give only the application to orthogonal polynomials.

We shall now state more precisely what types of recurrences we shall consider. In what follows, the reader who is interested only in real recurrences, loses nothing by thinking about the lattice as being $\mathbb{R}$. Recall that a Banach lattice is a complete normed space, which is also a partially ordered set, and in which the maximum of any two elements exists and the maximum operation is continuous. Recall that an order continuous lattice is a Banach lattice in which every increasing sequence which is bounded above converges. We refer to [5] for properties of such lattices. We shall state our results for an order continuous lattice L.

Let $f_n: L^m \to L$ be a sequence of functions and let $k_i \in \mathbb{Z} \setminus \{0\}$ for $i = 1, \ldots, m$. We consider recurrences of the type:

$$x_n = f_n(x_{n-k_1}, \ldots, x_{n-k_m}) \tag{1.1}$$

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(where the x_n are specified for negative values of n, that is, for $1-\max\{k_i \mid k_i>0\ i=1,..., m\} \leqslant n \leqslant 0$). We are interested in whether there is a solution (x_n) so that $x_n \geqslant 0$ for all $n \geqslant 1$ and whether such a solution is unique. Theorem 2.1 gives an answer to this question.

A typical example is whether there is a unique sequence $(a_n)_{n \geqslant 1}$ of positive reals satisfying,

$$\frac{n}{a_n} = a_{n+1} + a_n + a_{n-1} \qquad (\text{with} \quad a_0 = 0). \tag{1.2}$$

More generally letting $k \in \mathbb{R}$ a solution of the recurrence,

$$\frac{n}{a_n} = a_{n+1} + a_n + a_{n-1} + k \qquad \text{for} \quad n \geqslant 1,\ a_0 = 0 \tag{1.3}$$

with the property that $a_n > 0$ for all $n \geqslant 1$ is known to characterize those sequences of orthogonal polynomials $(p_n(x))_{n \geqslant 1}$ for which $p_n'(x)$ is a linear combination of two terms of the sequences $(p_n(x))_{n \geqslant 1}$ (see [2]). We shall consider more general recurrences in Theorem 2.2 and show they have unique positive solutions.

Our approach to these types of recurrences is to study certain equivalent fixed point problems. In [4] recurrences of the type

$$\frac{\gamma_n}{a_n} = a_{n+1} + a_n + a_{n-1} \qquad \text{for} \qquad n \geqslant 1,\ \gamma_n > 0,\ a_0 = 0 \tag{1.4}$$

were also investigated by using fixed points of certain operators. However, the techniques of [4] and this paper are somewhat different.

We are indebted to Paul Nevai for pointing out the paper [4] of Lew and Quarles and for mentioning the recurrences which arose in studying orthogonal polynomials.

2. Existence and Uniqueness of Solutions

Let L be an order continuous Banach lattice. Let $f_n: L^m \to L$ be a sequence of functions and let $k_i \in \mathbb{Z} \backslash \{0\}$ for $i=1,..., m$. We consider recurrences of the type,

$$x_n = f_n(x_{n-k_1},..., x_{n-k_m}) \qquad \text{for} \quad n \geqslant 1 \tag{2.1}$$

with x_n specified for $0 \geqslant n \geqslant 1-\max\{k_i \mid k_i>0,\ 1 \leqslant i \leqslant m\}$. Denote by $L_+^m = \{(x_1,..., x_m) \in L^m \mid x_i \geqslant 0$ for $1 \leqslant i \leqslant m\}$. A function $f: L_+^m \to L$ will be called positive if $f(L_+^m) \subseteq L_+$ and decreasing if $a \leqslant b$ for $a, b, \in L_+^m$ implies

$f(a) \geqslant f(b)$ ($a \leqslant b$ means each coordinate of a is less than each coordinate of b). We can now give a condition for a unique positive solution to (2.1).

THEOREM 2.1. (a) *If $f_n: L^m \to L$ is continuous, positive and decreasing on L^m_+, then there is a positive solution to* (2.1).

(b) *If for any two positive solutions $(x_n)_{n \geqslant 1}$, $(y_n)_{n \geqslant 1}$*

$$\|x_n - y_n\| \leqslant \lambda_n (\|x_{n-1} - y_{n-1}\| + \|x_{n+1} - y_{n+1}\|) \tag{2.2}$$

where,

$$0 \leqslant \lambda_n \leqslant \tfrac{1}{2} \quad \text{and} \quad \overline{\lim} \quad \lambda_n < \tfrac{1}{2} \tag{2.3}$$

$$\frac{t_{n+1}}{t_n} \to 1 \quad \text{where} \quad t_n = \|f_n(0, 0, ..., 0)\| \neq 0 \tag{2.4}$$

then there is a unique positive solution to (2.1).

Before proving Theorem 2.1 we point out an application to some recurrences which arise in approximation theory. Theorem 2.2 generalizes some results of Nevai [6], Bonan and Nevai [2] and Lew and Quarles [4].

THEOREM 2.2. *Let $\gamma_n \geqslant 0$, $\beta_n \in \mathbb{R}$ be given for $n \geqslant 1$. Consider*

$$\frac{\gamma_n^2}{a_n} = a_{n+1} + a_n + a_{n-1} + \beta_n \qquad \text{for} \quad n \geqslant 1, a_0 = a \in \mathbb{R}. \tag{2.5}$$

(a) *A positive solution to* (2.5) *exists.*

(b) *If $\beta_n = o(\gamma_n)$ and $(\gamma_{n+1}/\gamma_n) \to l \neq 0$ then $(a_n/\gamma_n) \to (l + l^{-1} + 1)^{-1/2}$ for any nonnegative solution (a_n).*

(c) *If $\beta_n = o(\gamma_n)$, $\beta_n \geqslant 0$ for all n, $a_0 \geqslant 0$ and $\gamma_{n+1}/\gamma_n \to 1$, then the solution is unique. In particular*

$$\frac{n}{c\, a_n} = a_{n+1} + a_n + a_{n-1} + k \qquad \text{for} \quad n \geqslant 1, a_0 = a \tag{2.6}$$

has a unique positive solution for $\{c, a, k\} \subseteq \mathbb{R}_+$.

Proof. (a) Let $f(x) = -x + (1 + x^2)^{1/2}$ and $g_n(x, y) = (x + y + \beta_n)/2\gamma_n$. Then (2.5) is the same as $a_n = \gamma_n f(g(a_{n+1}, a_{n-1}))$ if a_n is to be positive. By Theorem 2.1 (a) the result follows since f and g_n are continuous and f is positive and decreasing.

(b) We present an argument based on an idea of Freud (see [4]).

Note that for $n \geqslant 2$, $0 \leqslant a_n/\gamma_n \leqslant f(g_n(0,0))$. Moreover $f(g_n(0,0)) \to 1$ since $\beta_n = o(\gamma_n)$. So $v = \underline{\lim}\, a_n/\gamma_n$ and $u = \overline{\lim}\, a_n/\gamma_n$ exist. (2.5) is the same as

$$1 = \frac{a_n}{\gamma_n}\left[\frac{a_{n+1}}{\gamma_{n+1}}\frac{\gamma_{n+1}}{\gamma_n} + \frac{a_{n-1}}{\gamma_{n-1}}\frac{\gamma_{n-1}}{\gamma_n} + \frac{a_n}{\gamma_n} + \frac{\beta_n}{\gamma_n}\right] \tag{2.7}$$

Since $\beta_n = o(\gamma_n)$ we have

$$1 \leqslant v\left(u\,l + \frac{u}{l} + v\right) \tag{2.8}$$

and

$$1 \geqslant u\left(v\,l + \frac{v}{l} + u\right). \tag{2.9}$$

So by (2.8) and (2.9) $v \geqslant u$ and so a_n/γ_n has a limit L. Again by (2.7) we have

$$1 = L\left(L\,l + \frac{L}{l} + L\right) \tag{2.10}$$

and so $L = (l + l^{-1} + 1)^{-1/2}$.

(c) We show that the conditions of Theorem 2.1 (b) are fulfilled. Since $\gamma_{n+1}/\gamma_n \to 1$ it follows that $t_n = \gamma_n f(g_n(0,0))$ satisfies $t_{n+1}/t_n \to 1$. Now let $(x_n)_{n \geqslant 1}$ and $(y_n)_{n \geqslant 1}$ be any two positive solutions. Then,

$$\begin{aligned} |x_n - y_n| &= \gamma_n \left| f\left(\frac{x_{n+1} + x_{n-1} + \beta_n}{2\gamma_n}\right) - f\left(\frac{y_{n+1} + y_{n-1} + \beta_n}{2\gamma_n}\right)\right| \\ &\leqslant \frac{|f'(\xi_n)|}{2}\,[|x_{n+1} - y_{n+1}| + |x_{n-1} - y_{n-1}|] \end{aligned} \tag{2.11}$$

by the mean value theorem and the triangle inequality, for ξ_n between $(x_{n+1} + x_{n-1} + \beta_n)/2\gamma_n$ and $(y_{n+1} + y_{n-1} + \beta_n)/2\gamma_n$. Since $\beta_n \geqslant 0$, $x_n \geqslant 0$, $y_n \geqslant 0$, $a_0 \geqslant 0$ we have $\xi_n \geqslant 0$. Let $\lambda_n = |f'(\xi_n)|/2$. Since $|f'(x)| = 1 - x/((1+x^2)^{1/2})$ is decreasing and $|f'(0)| = 1$ we have $\lambda_n \leqslant \frac{1}{2}$ for all n. Also by Theorem 2.2 (b), $x_n/\gamma_n \to 1/\sqrt{3}$ and $y_n/\gamma_n \to 1/\sqrt{3}$ and so $\lambda_n \to \frac{1}{4}$. Therefore by Theorem 2.1 (b) there is a unique positive solution. ∎

We now prove Theorem 2.1.

Proof (*of Theorem* 2.1). (a) Let $\pi_n \colon L^{\mathbb{N}} \to L$ be the projection onto the nth coordinate. Put the natural ordering on $L^{\mathbb{N}}$ (that is $a \geqslant b$ if

$\pi_n(a) \geqslant \pi_n(b)$ for all n) and endow it with the product topology. Define $T: L^{\mathbb{N}} \to L^{\mathbb{N}}$ by

$$\pi_n(Tx) = f_n(x_{n-k_1}, \ldots, x_{n-k_m}) \qquad \text{for} \quad n \geqslant 1 \tag{2.12}$$

where $x = (x_n)_{n \geqslant 1}$. Because of the initial conditions on (2.1), T is well defined. Denote $\vec{0} = (0, 0, \ldots)$ and let $X = \prod_{n=1}^{\infty} [0, \pi_n T\vec{0}]$. Note that X is convex and compact by the Tychonoff theorem. Since f_n are positive and decreasing on L_+^m, $T(X) \subseteq X$. T is continuous since the f_n are continuous on L^m. Since $L^{\mathbb{N}}$ is locally convex, by the Schauder Tychonoff theorem (see [3]) T has a fixed point in X. This means (2.1) has a positive solution.

(b) Let $f: \mathbb{R}_+ \to \mathbb{R}$ be a function with the properties that $f \geqslant 0$, f is decreasing, $f(n+1)/f(n) \to 1$ (for $n \in \mathbb{N}$) and $f \in L_1(1, \infty)$. Choose N_1 large enough so that $\lambda_n \leqslant \lambda < \frac{1}{2}$ for $n \geqslant N_1$. Fix $0 < \varepsilon < (1/2\lambda) - 1$. Choose N_2 such that

$$\left.\begin{aligned} \frac{f(n-1)}{t_{n-1}} &< (1+\varepsilon)\frac{f(n)}{t_n} \\ \frac{f(n+1)}{t_{n+1}} &< (1+\varepsilon)\frac{f(n)}{t_n} \end{aligned}\right\} \quad \text{for} \quad n \geqslant N_2. \tag{2.13}$$

Let $N = \max(N_1, N_2)$. Choose $g: \mathbb{R}_+ \to \mathbb{R}$ with the properties on $[1, N]$: that g is positive, decreasing, concave, and satisfies:

$$\frac{f(N)}{t_N} \geqslant g(N-1) \tag{2.14}$$

$$g(N-1) \geqslant \frac{1}{2} g(N-2) + \lambda \frac{f(N)}{t_N}. \tag{2.15}$$

Define a sequence of real numbers $(a_n)_{n \geqslant 1}$:

$$a_n = \begin{cases} g(n)\, t_n & \text{for} \quad 1 \leqslant n \leqslant N-1 \\ f(n) & \text{for} \quad n \geqslant N. \end{cases} \tag{2.16}$$

On X define the metric,

$$d((x_n), (y_n)) = \sum_{n \geqslant 1} \frac{a_n}{t_n} \|x_n - y_n\|. \tag{2.17}$$

Note that since f is decreasing, positive, in $L_1(1, \infty)$ and $\|x_n - y_n\|/t_n \leqslant 2$ for (x_n), $(y_n) \in X$, $d((x_n), (y_n)) < \infty$. It is easily checked that the sequence (a_n) has the properties:

$$\frac{a_1}{t_1} \geqslant \lambda_2 \frac{a_2}{t_2} \tag{2.18}$$

$$\frac{a_n}{t_n} > \lambda_{n-1} \frac{a_{n-1}}{t_{n-1}} + \lambda_{n+1} \frac{a_{n+1}}{t_{n+1}} \quad \text{for} \quad n \geqslant N \tag{2.19}$$

$$\frac{a_n}{t_n} \geqslant \lambda_{n-1} \frac{a_{n-1}}{t_{n-1}} + \lambda_{n+1} \frac{a_{n+1}}{t_{n+1}} \quad \text{for} \quad 2 \leqslant n \leqslant N-1. \tag{2.20}$$

Suppose $x = (x_n)$ and $y = (y_n)$ are two positive solutions of (2.1). We show $d(x, y) < d(x, y)$ providing a contradiction. We have,

$$\begin{aligned}
d(x, y) &= \sum_{n \geqslant 1} \frac{a_n}{t_n} \|x_n - y_n\| \\
(\text{by } 2.2) &\leqslant \sum_{n \geqslant 1} \frac{a_n \lambda_n}{t_n} [\|x_{n+1} - y_{n+1}\| + \|x_{n-1} - y_{n-1}\|] \\
(\text{since } x_0 = y_0) &= \sum_{n \geqslant 2} \left(\frac{a_{n-1} \lambda_{n-1}}{t_{n-1}} + \frac{a_{n+1} \lambda_{n+1}}{t_{n+1}} \right) \|x_n - y_n\| + \frac{a_2 \lambda_2}{t_2} \|x_1 - y_1\| \\
&< \sum_{n \geqslant 1} \frac{a_n}{t_n} \|x_n - y_n\| \qquad \text{by (2.18), (2.19), and (2.20)} \\
&= d(x, y). \quad \blacksquare
\end{aligned}$$

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